Key

Math 4 Name________________

**2-3 Solving Systems with Matrices** Date__________

Learning Goals:

- *I can use determinants to find whether a matrix is invertible.*
- *I can use matrices to solve systems of linear equations.*
- *I can use matrices to encode and decode messages.*

I. In Lesson 2-2 you learned about properties of matrices. In this lesson we will further develop the idea of the **inverse of a matrix**.

A. If a matrix has an inverse, it is called **invertible** (or **nonsingular**); otherwise, it is called **singular.** Remember, only a square matrix can have an inverse. But not all square matrices have inverses. If, however, a matrix does have an inverse, that inverse is unique.

1. To determine if a matrix has an inverse, we can use something called its **determinant**. The determinant of a $2 \times 2$ matrix is defined as follows:

**Definition of the Determinant of a 2 × 2 Matrix**

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of the matrix is given by:

$$\det(A) = |A| = ad - bc$$

2. Given $A = \begin{bmatrix} -8 & 5 \\ -3 & 4 \end{bmatrix}$. Use the formula above to calculate $|A|$.

$\det(A) = |A| = (-8)(4) - (-3)(5) = -32 + 15 = \boxed{-17}$

*****If the determinant of matrix ≠ 0, then the matrix has an inverse.**

Based on your computations, does $A$ have an inverse? Yes!

B. To find the inverse of a $2 \times 2$ matrix, we use the following formula:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then, } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

1. Looking at this formula, explain why it makes sense that a nonsingular matrix must have a nonzero determinant.

If the determinant was zero, the fraction in front would be $\frac{1}{0}$.

2. Use the above formula to find $A^{-1}$ (for #2, part A).

$$= \frac{1}{-17}\begin{bmatrix} 4 & -5 \\ 3 & -8 \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{4}{17} & \frac{5}{17} \\ -\frac{3}{17} & \frac{8}{17} \end{bmatrix}}$$

3. Now use your calculator to verify what you found in #2.

Verified!

II. Why is the inverse of a matrix so important? You will see in this next section . . . . .

A. Solve the system of equations below using the method of your choosing.

$$\begin{cases} 3x + y = 33 \\ 5x - 8y = 258 \end{cases} \longrightarrow y = 33 - 3x$$

$5x - 8(33 - 3x) = 258$

$5x - 264 + 24x = 258$

$29x - 264 = 258$

$29x = 522 \quad x = 18$

$y = 33 - 3(18)$

$y = -21$

$\boxed{(18, -21)}$

B. Solving a system using the Inverse-Matrix Method:

In previous courses you learned multiple methods of solving a system of equation: graphically, substitution, and linear combinations. The Inverse-Matrix Method is yet another process.

1. The Inverse-Matrix Method Matrix uses matrix multiplication to represent a system of linear equations. Note how the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

$x_1 = x$
$x_2 = y$
$x_3 = z$

can be written as the matrix equation $AX = B$ where $A$ is the *coefficient matrix* of the system, and $X$ and $B$ are column matrices.

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \times \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

2. Solving a system using matrices boils down to setting up and solving a matrix equation. Note the similarity to solving a linear equation to that of a matrix equation:

| **Solving a Linear Equation** | **Solving a Matrix Equation** |
|---|---|
| $3x = 6$ | $AX = B$ |
| $\frac{1}{3}$ ← (*inverse of* 3) × 3*x* = (*inverse of* 3) × 6 | (*inverse of A*) × *AX* = (*inverse of A*) × *B* |
| *x* = (*inverse of* 3) × 6 | *X* = (*inverse of A*) × *B* |
| $x = \frac{1}{3} \times 6$ | $X = A^{-1} \times B$ |

3. Let's see how this works. Study the following example:
Consider the following system of linear equations.

$$\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ \quad\;\; x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$$

a. Write this system as a **matrix equation**. $AX = B.$

In matrix form, $AX = B$, the system can be written as follows.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

$$A \bullet X = B$$

**Solution Matrix**

b. Solve the matrix equation for $X$. $X = A^{-1}B.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}^{-1} \bullet \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} \quad \therefore \quad \boxed{X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.}$$

$$X = A^{-1} \bullet B$$

4. Solve the following system using the Inverse-Matrix Method.

$$\begin{cases} -x + y - z = -14 \\ 2x - y + z = 21 \\ 3x + 2y + z = 19 \end{cases}$$

a. Write the system as a matrix equation.

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -14 \\ 21 \\ 19 \end{bmatrix}$$

$$A \bullet X = B$$

b. Use your calculator to solve the matrix equation for $X$.

Solution Matrix: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 4 \end{bmatrix}$ So, $x = 7$, $y = -3$, $z = 4$

*****2-3 Homework is on the next page!!**

**Lesson 2-3 Homework**

****Please show your work on a separate piece of paper.*

1. Without using a calculator, show that $B$ is the inverse of $A$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

2. Without using a calculator, find the inverse of each matrix (if it exists).

$$A = \begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$$

***Use your calculator for the remaining problems.***

3. Use the Inverse-Matrix Method to solve the following systems (if possible).
*You must write the matrix equation as well as your final solution matrix.*

a. $\begin{cases} 4x - 3y = -5 \\ -x + 3y = 12 \end{cases}$

b. $\begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases}$

c. $\begin{cases} -x + y - z = -14 \\ 2x - y + z = 21 \\ 3x + 2y + z = 19 \end{cases}$

d. $\begin{cases} x + 2y + 2z + 4w = 11 \\ 3x + 6y + 5z + 12w = 30 \\ x + 3y - 3z + 2w = -5 \\ 6x - y - z + w = -9 \end{cases}$

e. $\begin{cases} -x + y = -22 \\ 3x + 4y = 4 \\ 4x - 8y = 32 \end{cases}$

f. $\begin{cases} x_1 - 5x_2 + 2x_3 = -20 \\ -3x_1 + x_2 - x_3 = 8 \\ -2x_2 + 5x_3 = -16 \end{cases}$

Use the Inverse-Matrix Method to solve the following problems.

4. A serving of roast beef has 17 grams of protein and 11 milligrams of calcium. A serving of mashed potatoes has 2 grams of protein and 25 milligrams of calcium. How many servings of each are needed to get 40 grams of protein and 97 milligrams of calcium?

$\boxed{M4}$ 2-3 Homework

1.) $|A| = 1\cdot4 - 2\cdot3 = 4-6 = -2$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = B \checkmark$$

2.) $|A| = -12\cdot-2 - 3\cdot5 = 24-15 = \boxed{9}$; $A^{-1} = \frac{1}{9}\begin{bmatrix} 2 & -3 \\ -5 & -12 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{2}{9} & -\frac{1}{3} \\ -\frac{5}{9} & -\frac{4}{3} \end{bmatrix}}$

$|B| = (-4)(3) - (-6)(2) = -12+12 = \boxed{0}$ **No inverse! B is singular.**

CHALLENGE!!

$|C| = \sin\theta\cdot\sin\theta - \cos\theta\cdot -\cos\theta = \sin^2\theta + \cos^2\theta = \boxed{1}$

Trig identity

$$\boxed{C^{-1} = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}}$$

3.) a) $\begin{bmatrix} 4 & -3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 12 \end{bmatrix}$ Solution $= \boxed{\begin{bmatrix} 7/3 \\ 43/9 \end{bmatrix}}$

b) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **No solution**

c) See Page 3 solution.

d) $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 6 & 5 & 12 \\ 1 & 3 & -3 & 2 \\ 6 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 11 \\ 30 \\ -5 \\ -9 \end{bmatrix}$ Solution $= \boxed{\begin{bmatrix} -1 \\ 1 \\ 3 \\ 1 \end{bmatrix}}$

(#3 continued)

e) $\begin{bmatrix} -1 & 1 & 0 \\ 3 & 4 & 0 \\ 4 & -8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix}$ $\boxed{\text{No solution}}$

f) $\begin{bmatrix} 1 & -5 & 2 \\ -3 & 1 & -1 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20 \\ 8 \\ -16 \end{bmatrix}$ Solution $= \boxed{\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}}$

4.) $17r + 2m = 40$
$11r + 25m = 97$

$\begin{bmatrix} 17 & 2 \\ 11 & 25 \end{bmatrix} \begin{bmatrix} r \\ m \end{bmatrix} = \begin{bmatrix} 40 \\ 97 \end{bmatrix}$ Solution $= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

2 servings of roast beef
3 servings of mashed potatoes